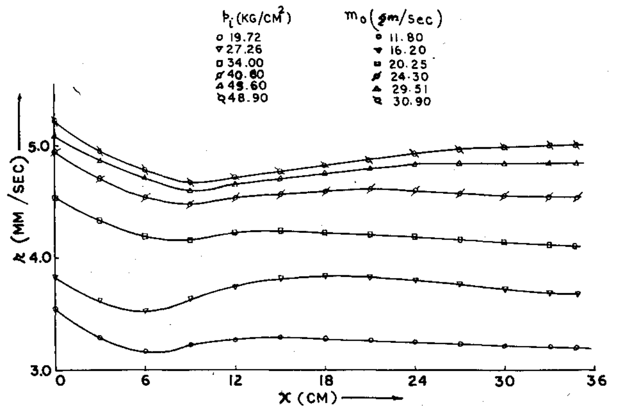

Fig. 3 Axial regression rate variation with oxidizer mass-flow rate.

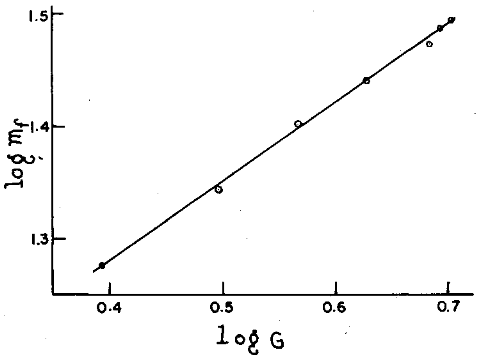

Fig. 4 Fuel mass consumption rate dependence on total mass flux.

suggest that, by a suitable selection of nozzle, a wide range of nominal thrust levels can be achieved in a hybrid engine.

The variation of local regression rate along the axial position of the fuel grain is presented in Fig. 3 for different oxidizer mass-flow rates. Higher oxidizer mass-flow rate is found to yield higher regression rate. An increased oxidizer mass-flow rate is bound to yield increased value of regression rate because the regression rate of a hybrid fuel is governed by the mass flux.

Figure 4 depicts the dependence of fuel mass consumption rate on total mass flux. The fuel mass consumption rate is found to have a power law variation with total mass flux as expected[1,2] as higher oxidizer mass flux obviously will result in higher regression rate and will be reflected in higher fuel mass consumption rate.

The investigation was discontinued for injection pressures below 17 kg/cm^2, i.e., for oxidizer mass-flow rate less than 10 g/sec for reasons of incomplete combustion. Examination of the fuel grain after such test runs revealed relatively thick char deposition indicating inability of fuel to pyrolyze and decompose completely by the heat flux of the flame zone.

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Velocity of Bodies Powered by Polyatomic Cold-Gas Thrusters

M.D. Bennett*

Sandia Laboratories, Albuquerque, N. Mex.

Nomenclature

b = mass-fraction parameter $= \lambda/(1-\lambda)$
c_i = initial sound speed in propellant gas, m/sec
I_s = specific impulse of thruster, m/sec
k = specific-heats parameter $= 2/(\gamma - 1)$
m_f = vehicle final mass, kg
m_i = vehicle initial mass, including propellant gas, kg
S = thruster nozzle throat area, m^2
t = time, sec
V = thruster reservoir volume, m^3
v = vehicle translational velocity increment, m/sec
β = specific-heats term $= [(\gamma-1)/2][(\gamma+1)/2]^{-(\zeta/2)}$
γ = ratio of specific heats of propellant gas
ζ = specific-heats factor $= (\gamma+1)/(\gamma-1)$
θ = specific-heats term $= [2/(\gamma+1)][2/(\gamma-1)]^{1/2}$
λ = vehicle mass fraction $= m_f/m_i$
τ = normalized time $= 1 - [1/(1+\psi t)]$
ψ = configuration parameter $= \beta S c_i / V$, sec^{-1}

Introduction

AN integral equation was derived in Ref. 1 which expresses the ideal velocity imparted to a missile as a result of discharging inert propulsive gases from thermally insulated tanks. The analytical solution pertaining to the general case of arbitrary specific-heats ratio γ had the form of an infinite series. However, for the category of fluids that yield integer values of the function $f(\gamma) = 2/(\gamma-1)$, which describes the molecular degrees of freedom in an ideal gas, a relatively compact formula evolved. Ultimately, the special solution was expanded in detail for situations involving the most elementary class of fluids in the category – the monatomic gases.

Expansion of the special solution was extended in Ref. 2 to include the diatomic gases. In order to complete the spectrum of working fluids for cold-gas thrusters, the polyatomic gases now are considered. Since the algebraic expressions of velocity become somewhat unwieldy with the many-degrees-of-freedom gases, numerical evaluations of the integral equation have been performed. These calculations are summarized and compared with the previous information for monatomic and diatomic gases in the following discussion.

Integral Equation Describing Velocity

If the thermodynamic properties of the fluid inside the thruster reservior maintain a uniform spatial distribution during the discharge process, the time-dependent gasdynamic equations can be integrated analytically to give the pressure and temperature decay histories, provided that the flow is

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*Member of Technical Staff, Aerodynamics Department. Associate Fellow AIAA.

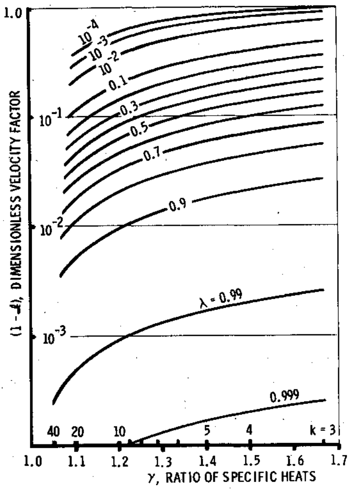

Fig. 1 Terminal velocity parameter as a function of specific-heats ratio showing effect of propellant quantity.

assumed to be one-dimensional, isentropic, and quasisteady, with no thermal radiation. In accordance with that model, and for the case of choked flow conditions at the nozzle throat, and with complete expansion of the exhaust, the equation describing translational motion in a gravitationless vacuum takes the form [see Eq. (7), Ref. 1]

$$v = \zeta I_s(\tau - \mathcal{I}_\tau) \tag{1}$$

where $\zeta = (\gamma + 1)/(\gamma - 1)$ denotes a factor determined by the specific-heats ratio, $I_s = \theta c_i$ represents the thruster specific impulse, and τ and $\mathcal{I}_\tau$ are the normalized time and a definite integral involving the time, respectively.

The dimensionless integral $\mathcal{I}_\tau$ in Eq. (1) previously was found to be

$$\mathcal{I}_\tau = b^{(1/k)} \int_{\xi_1}^{\xi_2} \frac{d\xi}{1 + \xi^k} \tag{2}$$

where the coefficient $b^{(1/k)} = [\lambda/(1-\lambda)]^{(1/k)}$ is a known constant related to the mass fraction of the propellant, with the exponent $1/k = (\gamma - 1)/2$ determined by the specific-heats ratio. Equation (2) has the lower and upper limits of integration $\xi_1 = (1-\tau)(1/b)^{(1/k)}$ and $\xi_2 = (1/b)^{(1/k)}$, respectively.

In the absence of forces other than the reaction due to releasing the compressed gas, the maximum velocity occurs at time $t = \infty$. In the terminal state of motion, Eq. (1) becomes $v_{\max} = \zeta I_s (1 - \mathcal{I})$, or, written in terms of the initial speed of sound c_i in the working fluid,

$$v_{\max} = k^{3/2}(1 - \mathcal{I})c_i \tag{3}$$

The integral $\mathcal{I} \equiv (\mathcal{I}_\tau)_{\tau=1}$ in Eq. (3) is identical to Eq. (2) except that the lower integration limit reduces to zero when $\tau = 1$; i.e.,

$$\mathcal{I} = b^{(1/k)} \int_0^{(1/b)^{(1/k)}} \frac{d\xi}{1 + \xi^k} \tag{4}$$

Once this integral has been evaluated, the final velocity of the variable mass body may be determined readily by means of Eq. (3).

Solutions of the Integral Equation

The general solution of Eq. (2) or (4) involves an infinite geometric series [Eq. (9), Ref. 1] if the exponent k of the variable in the integrand has arbitrary value. In the particular case in which k may be specified as a positive integer, a finite-series solution [Eq. (10), Ref. 1] exists,† but the terms of the resulting equation become increasingly cumbersome, as well as more in number, as k grows large. For the polyatomic gases, which are characterized by the larger k's, a numerical integration technique[3] was employed. By that process, evaluations of Eq. (4) were acquired with k ranging up to 20. The results of the calculations are summarized in Fig. 1, along with the previous data for monatomic ($k = 3$) and diatomic ($k = 5$) gases. It may be seen that, as k decreases or, alternately, as γ increases, the integral factor $(1 - \mathcal{I})$ that appears in the velocity equation continually increases for all values of λ. The magnitude of the factor, rounded to four significant figures, also is shown in Table 1 for selected values of k.

†According to the kinetic theory of gases for an elementary molecular model, the number of internal degrees of freedom n is related to the ratio of specific heats by the expression $n = 2/(\gamma - 1)$. Since the parameter $k = 2/(\gamma - 1)$ has an identical definition, certain integer values of k are equivalent to the degrees of freedom in an ideal gas. Consequently, the k-integer solution pertains to a rather comprehensive catalog of fluids.

Table 1 Dimensionless velocity factor $(1 - \mathcal{I})$ for several polyatomic cold-gas thrusters as a function of propellant mass fraction ($\lambda = m_f/m_i$)

λ	$k = 6$ ($\gamma = 1.333$)	$k = 8$ ($\gamma = 1.25$)	$k = 10$ ($\gamma = 1.2$)	$k = 20$ ($\gamma = 1.1$)
0	1	1	1	1
10^{-8}	0.9514	0.8974	0.8389	0.6003
10^{-7}	0.9287	0.8632	0.7972	0.5515
10^{-6}	0.8953	0.8175	0.7446	0.4967
10^{-5}	0.8463	0.7567	0.6785	0.4353
10^{-4}	0.7744	0.6755	0.5953	0.3664
10^{-3}	0.6690	0.5674	0.4905	0.2892
10^{-2}	0.5151	0.4237	0.3590	0.2025
0.1	0.2951	0.2354	0.1956	0.1059
0.2	0.2139	0.1692	0.1399	0.07490
0.3	0.1633	0.1285	0.1060	0.05639
0.4	0.1259	0.09884	0.08133	0.04310
0.5	0.09623	0.07535	0.06191	0.03271
0.6	0.07148	0.05587	0.04585	0.02417
0.7	0.05023	0.03920	0.03214	0.01691
0.8	0.03160	0.02463	0.02018	0.01060
0.9	0.01499	0.01167	0.009554	0.005010
0.99	0.001435	0.001116	0.0009134	0.0004785
0.999	0.0001429	0.0001112	0.00009095	0.00004764
0.9999	0.00001429	0.00001111	0.000009091	0.000004762
1	0	0	0	0

The velocity history of any particular system may be obtained from either the analytical solutions in Ref. 1 or numerical integration of Eq. (2). For engineering purposes, a more convenient procedure prevails when the propellant comprises a small part of the total mass, which often is the arrangement. In that event, velocity becomes directly proportional to the impulse expenditure. The necessary knowledge of the impulse history can be derived from Eqs. (1-3) of Ref. 1. It is found from those equations that the time required to deliver some fraction q of the total impulse is $\tau = 1-(1-q)^{(1/\zeta)}$, or, in terms of real time t,

$$\psi t = 1/(1-q)^{(1/\zeta)} - 1 \tag{5}$$

With a given thruster configuration parameter ψ, the variation of impulse may be determined immediately from Eq. (5). Once the impulse distribution has been established, the approximate velocity history becomes $v/v_{\max} = q(t)$, or

$$v/v_{\max} = 1 - [1/(1+\psi t)]^{\zeta} \quad \text{if} \quad \lambda \cong 1 \tag{6}$$

Using Eq. (5), it may be seen that for propulsive gases having, for example, seven active degrees of freedom, and with $\psi = 1$/sec, 90% of the available impulse is released in about ⅓ sec. An equivalent part of the terminal velocity occurs during that time interval if the mass faction of the working fluid is initially small.

With the preceding information, the ideal characteristics of an extensive group of cold-gas thrusters may be assessed readily.

Acknowledgment

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Optimal Terminal Guidance with Constraints at Final Time

Randy J. York*

Western Kentucky University, Bowling Green, Ky.

and

Harold L. Pastrick†

U.S. Army Missile Research and Development Command, Redstone Arsenal, Ala.

I. Introduction

RECENT intelligence suggests that the impenetrable nature of heavy armor may be susceptible to missile attacks at a relatively high angle of impact, with respect to the horizon. In many modes of direct encounter, the target may not be reachable with a body pitch attitude angle of the proper magnitude. There are several possible reasons for this condition, including lack of energy (fuel), lack of time to maneuver into the more desirable attitude, or lack of control information by appropriate sensors to command the response. This condition has been recognized for some time at the Missile Research and Development Command, and consequently there have been attempts to modify trajectory shapes by a variety of predetermined control laws. However, there has been a certain lack of robustness in the solutions obtained over the entire range of conditions anticipated. This situation motivated a search for optimal solutions to the guidance problem and a study of tradeoffs among the suboptimal candidates that were deemed feasible.

Terminal guidance schemes for tactical missiles may be based on a classical approach, such as a proportional navigation and guidance law,[1,2] or on a modern control theoretic approach.[3-5] In the latter, a control law is derived in terms of time-varying feedback gains when formulated as a linear quadratic control problem. A suboptimal terminal guidance system for re-entry vehicles, derived using the modern approach, was the basis for the initial work on this problem.

Kim and Grider[6] studied a suboptimal terminal guidance system for a re-entry vehicle by placing a constraint on the body attitude angle at impact. Their problem was oriented to a long-range, high-altitude mission. Their scenario was formulated as a linear quadratic control problem with certain key assumptions. The angle of attack of the re-entry vehicle was assumed to be small and thus was neglected. Furthermore, the autopilot response was assumed to be instantaneous, i.e., with no lag time attributed to the transfer of input commands to output reaction.

These conditions have been studied in an extension of their earlier work.[7] A formulation is given for a system that has finite time delay. In fact, the increase and decrease in time delay has interesting ramifications on the solution. The angle-of-attack assumption is investigated, and, although not solved analytically in closed form, the system is derived.

There is more than just a passing academic interest in this problem. As suggested previously, the antiarmor role of several Army weapon systems very well may be enhanced by this technique. The reduction to a practical implementation or mechanization will be studied and described in a future paper. This paper, however, summarizes the feasibility of the concept.

II. State Representation and Problem Formulation

The geometry of the tactical missile-target position is given in Fig. 1. Assume that the angle of attack is small and thus can be neglected (this assumption will be considered later), and choose the following set of variables:

$$x = \begin{bmatrix} Y_d \\ \dot{Y}_d \\ A_L \\ \theta \end{bmatrix} \equiv \begin{bmatrix} Y_t - Y_m \\ \dot{Y}_t - \dot{Y}_m \\ A_L \\ \theta \end{bmatrix} \tag{1}$$

where Y_d is the position variable from the missile to the target projected on the ground; Y_t is the position variable of the target; Y_m is the position variable of the missile projected on the ground; $\dot{Y}_d$ is the derivative of Y_d, the missile to the target velocity projected on the ground; A_L is the lateral acceleration of the missile; θ is the body attitude angle of the missile; and α is the angle of attack of the missile shown in Fig. 1.

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*Associate Professor, Department of Mathematics and Computer Science.

†Research Aerospace Engineer, Guidance and Control Directorate. Member AIAA.